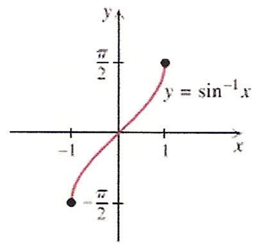
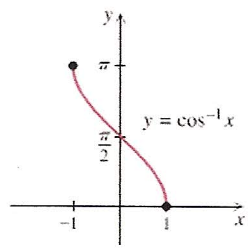
Math 1060 – 4.1 Notes

The Inverse Trigonometric Functions

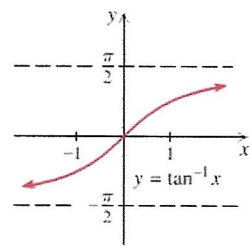
In order for a function to have an inverse, the function must be one-to-one. In other words, it must pass the horizontal line test. Since the trigonometric functions are periodic, we must restrict the domain so that they will pass the horizontal line test. Therefore, the inverse functions will have a restricted range.



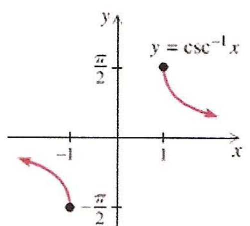
Domain $[-1, 1]$
Range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$



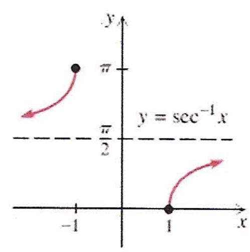
Domain $[-1, 1]$
Range $[0, \pi]$



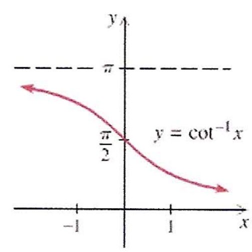
Domain $(-\infty, \infty)$
Range $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$



Domain $(-\infty, -1] \cup [1, \infty)$
Range $\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$



Domain $(-\infty, -1] \cup [1, \infty)$
Range $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$



Domain $(-\infty, \infty)$
Range $(0, \pi)$

The **inverse sine function** is sometimes called the **arc sine**, and is abbreviated $\arcsin(x)$ or $\sin^{-1}(x)$. Similarly, the other inverse functions are often called the **arc cosine** and **arc tangent**, abbreviated $\arccos(x)$ or $\cos^{-1}(x)$ and $\arctan(x)$ or $\tan^{-1} x$.

$\sin^{-1} x$ is the angle in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ whose sine is x.

$\cos^{-1}(x)$ is the angle in $[0, \pi]$ whose cosine is x.

$\tan^{-1}(x)$ is the angle in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ whose tangent is x.

Example: Find the exact value of each expression without using a table or calculator.

a) $\sin^{-1}\left(\frac{1}{2}\right) = \boxed{\pi/6}$

b) $\arcsin\left(-\frac{\sqrt{2}}{2}\right) = \boxed{-\pi/4}$

c) $\tan^{-1}(1) = \boxed{\pi/4}$

d) $\arccos\left(-\frac{1}{2}\right) = \boxed{2\pi/3}$

Example: Find the angle α.

a) $\sin\alpha = 0.56,\ -90° \le \alpha \le 90°$

$\sin^{-1}(0.56) = \boxed{34.1°}$

b) $\tan\alpha = -3,\ -\pi/2 < \alpha < \pi/2$

$\tan^{-1}(-3) = \boxed{-1.2 \text{ rad}}$

c) $\cos\alpha = 0.23,\ 0° \le \alpha \le 180°$

$\cos^{-1}(0.23) = \boxed{76.7°}$

d) $\cos\alpha = -0.82,\ 0 \le \alpha \le \pi$

$\cos^{-1}(-0.82) = \boxed{2.5 \text{ rad}}$

Inverses of the Reciprocal Trigonometric Functions

$\csc^{-1} x$ is the angle in $\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$ whose cosecant is x.

$\sec^{-1}(x)$ is the angle in $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$ whose secant is x.

$\cot^{-1}(x)$ is the angle in $(0, \pi)$ whose cotangent is x.

Identities

$$\csc^{-1}(x) = \sin^{-1}\left(\frac{1}{x}\right)$$

$$\sec^{-1}(x) = \cos^{-1}\left(\frac{1}{x}\right)$$

$$\cot^{-1}(x) = \begin{cases} \tan^{-1}(1/x) & \text{for } x > 0 \\ \tan^{-1}(1/x) + \pi & \text{for } x < 0 \\ \pi/2 & \text{for } x = 0 \end{cases}$$

$$\cot^{-1}(x) = \frac{\pi}{2} - \tan^{-1}(x)$$

Example: Find the exact value of each expression without using a table or calculator.

a) $\operatorname{arcsec}(-2)$ If the secant is -2, the cosine is $-\frac{1}{2}$

$\arccos\left(-\frac{1}{2}\right) = \boxed{\frac{2\pi}{3}}$

b) $\csc^{-1}(2)$ If csc is 2, sin is $\frac{1}{2}$

$\sin^{-1}\left(\frac{1}{2}\right) = \boxed{\frac{\pi}{6}}$

c) $\operatorname{arccot}\left(\frac{1}{\sqrt{3}}\right)$ If cot is $\frac{1}{\sqrt{3}}$, tan is $\sqrt{3}$

$\arctan(\sqrt{3}) = \boxed{\frac{\pi}{3}}$

Example: Find the approximate value of each expression rounded to 4 decimal places.

a) $\operatorname{arccsc}(-1.4713)$

$\arcsin\left(\frac{1}{-1.4713}\right) = \boxed{-.7473}$

b) $\cot^{-1}(-2.5)$

$\tan^{-1}\left(\frac{1}{-2.5}\right) = -.3805$

$\cot^{-1}$ must be in $(0, \pi)$, so add π. $-.3805 + \pi = \boxed{2.7611}$

c) $\sec^{-1}(4.328)$

$\cos^{-1}\left(\frac{1}{4.328}\right) = \boxed{1.3376}$

Example: Find the exact value of each composition.

a) $\sin\left(\cot^{-1}(-1)\right)$

$= \sin\left(\frac{3\pi}{4}\right)$

$= \boxed{\frac{\sqrt{2}}{2}}$

b) $\sin^{-1}\left(\sin\left(\frac{5\pi}{4}\right)\right)$

$= \sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$

$= \boxed{-\frac{\pi}{4}}$

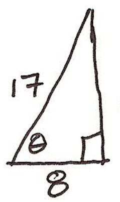
c) $\arcsin\left(\cos\left(\frac{\pi}{6}\right)\right)$

$= \arcsin\left(\frac{\sqrt{3}}{2}\right)$

$= \boxed{\frac{\pi}{3}}$

Examples: Find the exact value of each composition.

a) $\sin\left(\underbrace{\cos^{-1}\left(\frac{8}{17}\right)}_{\theta}\right)$ $\cos\theta = \frac{x}{r} = \frac{8}{17}$

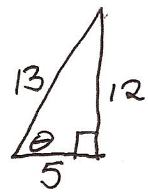


$y = \sqrt{17^2 - 8^2} = 15$

$\boxed{\sin\theta = \frac{15}{17}}$

b) $\cot\left(\underbrace{\arccos\left(\frac{5}{13}\right)}_{\theta}\right)$ $\cos\theta = \frac{x}{r} = \frac{5}{13}$

$\cot\theta = \frac{x}{y} = \boxed{\frac{5}{12}}$



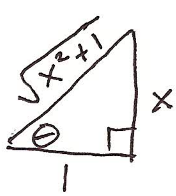
$y = \sqrt{13^2 - 5^2} = 12$

Example: Find an equivalent algebraic expression for $\sin\left(\underbrace{\arctan(x)}_{\theta}\right)$

$\arctan x = \theta$

$\tan\theta = x = \frac{x}{1}$

$\sin\theta = \boxed{\frac{x}{\sqrt{x^2+1}}}$



$r = \sqrt{x^2 + 1^2}$

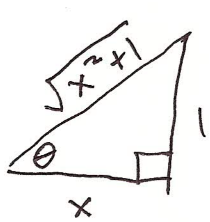
$= \sqrt{x^2 + 1}$

Example: Find an equivalent algebraic expression for $\sin\left(\underbrace{\operatorname{arccot}(x)}_{\theta}\right)$

$\operatorname{arccot} x = \theta$

$\cot\theta = \frac{x}{1}$

$\sin\theta = \boxed{\frac{1}{\sqrt{x^2+1}}}$



$r = \sqrt{x^2 + 1^2}$

$= \sqrt{x^2 + 1}$

Basic Sine, Cosine, and Tangent Equations

Basic steps for solving $\cos x = a$:

1. **Find all the angles on the unit circle (on $[0, 2\pi]$) that satisfy the equation.** One of these solutions will be $s = \cos^{-1} a$ and the other will be $2\pi - s = 2\pi - \cos^{-1} a$.
2. **Add or subtract multiples of 2π from each angle.**

Basic steps for solving $\sin x = a$:

1. **Find all the angles on the unit circle (on $[0, 2\pi]$) that satisfy the equation.** You can do this by looking at the unit circle (usually this is less confusing) or by working algebraically. If $s = \sin^{-1} a > 0$, one of these solutions will be $s = \sin^{-1} a$ and the other will be $\pi - s = \pi - \sin^{-1} a$. If $s = \sin^{-1} a < 0$, the two solutions are $s + 2\pi = \sin^{-1} a + 2\pi$ and $\pi - s = \pi - \sin^{-1} a$.
2. **Add or subtract multiples of 2π from each angle.**

Don't let the algebra freak you out! All you are doing is finding all the angles on the unit circle that satisfy the equation and adding $2k\pi$ to each one to form your solution set.

Basic steps for solving $\tan x = a$:

1. **Find one angle on the unit circle that satisfies the equation.** This will be either $s = \tan^{-1} a$ if this value is positive, or $s + \pi = \tan^{-1} a + \pi$ if $s = \tan^{-1} a$ is negative.
2. **Add or subtract multiples of π from each angle.** (Remember that the tangent repeats every π instead of every 2π like sine and cosine).

Examples: Find all real numbers that satisfy each equation.
radians

a) $\sin x = 1$

$\boxed{\{x \mid x = \frac{\pi}{2} + 2k\pi\}}$

b) $\cos x = 0$ $\quad \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}; \frac{7\pi}{2}$, etc.

$\boxed{\{x \mid x = \pi/2 + k\pi\}}$ or $\boxed{\{x \mid x = \pi/2 + 2k\pi \text{ or } x = 3\pi/2 + 2k\pi\}}$

c) $\cos x = -1/2$ $\quad 2\pi/3, 4\pi/3$

$\boxed{\{x \mid x = 2\pi/3 + 2k\pi \text{ or } x = 4\pi/3 + 2k\pi\}}$

d) $\sin x = \sqrt{2}/2$ $\quad \pi/4, 3\pi/4$

$\boxed{\{x \mid x = \pi/4 + 2k\pi \text{ or } x = 3\pi/4 + 2k\pi\}}$

e) $\tan x = -\sqrt{3}$ $\quad 2\pi/3$

$\boxed{\{x \mid x = 2\pi/3 + k\pi\}}$

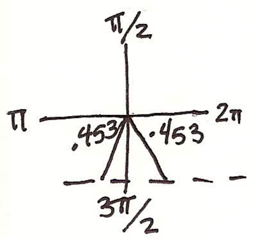
f) $\tan x = 1$ $\quad \pi/4$

$\boxed{\{x \mid x = \pi/4 + k\pi\}}$

g) $\sin x = -.4375$

$\sin^{-1}(-.4375) = -.453$

$-.453 + 2\pi = 5.830$



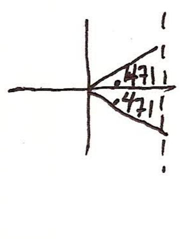
other angle with same sine is $\pi + .453 = 3.595$

$\boxed{\{x \mid x = 3.595 + 2k\pi \text{ or } x = 5.830 + 2k\pi\}}$

h) $\cos x = .8913$

$\cos^{-1}(.8913) = .471$

other angle with same cosine is $2\pi - .471 = 5.813$



$\boxed{\{x \mid x = .471 + 2k\pi \text{ or } x = 5.813 + 2k\pi\}}$

Examples: Find all angles in $[0°, 360°]$ that satisfy each equation

a) $\cos x = \sqrt{3}/2$

$\boxed{\{30°, 330°\}}$

b) $\tan x = -3.5$

tangent repeats every 180°

$\tan^{-1}(-3.5) = -74.1°$

$-74.1° + 180° = 105.9°$

$105.9° + 180° = 285.9°$

$\boxed{\{105.9°, 285.9°\}}$

Sometimes, you have to do a bit of algebra before you can use the techniques above.

a) Solve $2\sin\alpha - 1 = 0$ for $0 \le \alpha \le 2\pi$.

$2\sin\alpha = 1$

$\sin\alpha = 1/2$

$\boxed{\{\pi/6, 5\pi/6\}}$

b) Solve $3\sin(\beta) + 6 = 5\sin(\beta) + 7$ for $0° \le \beta \le 360°$.

$-3\sin(\beta) \quad -3\sin(\beta)$

$6 = 2\sin\beta + 7$

$-1 = 2\sin\beta$

$\sin\beta = -1/2$

$\boxed{\{7\pi/6, 11\pi/6\}}$

We can also solve trigonometric functions involving multiple variables for a specific variable by using the definitions of the inverse functions.

Example: Solve $b = 7\tan\left(\frac{a}{3}\right) - d$ for a where $-\frac{3\pi}{2} < a < \frac{3\pi}{2}$. ← This means $-\frac{\pi}{2} < \frac{a}{3} < \frac{\pi}{2}$, so when we take $\tan^{-1}$, we will get $\frac{a}{3}$. We don't have to add π.

$b + d = 7\tan\left(\frac{a}{3}\right)$

$\frac{b+d}{7} = \tan\left(\frac{a}{3}\right)$

$\frac{a}{3} = \tan^{-1}\left(\frac{b+d}{7}\right)$

$\boxed{a = 3\tan^{-1}\left(\frac{b+d}{7}\right)}$

Multiple Angle Equations

Often, equations involve expressions like $\sin 2x$, $\cos 3\alpha$, or $\tan(x/2)$, all of which involve multiples of the variable rather than a single variable. To solve these equations, we solve for the multiple variable just as we would solve for a single variable and then multiply or divide to get the single variable in the last step.

Example: Find all solutions in degrees to $\sin 2\alpha = \sqrt{3}/2$.

$$\sin 2\alpha = \frac{\sqrt{3}}{2}$$

$$\frac{2\alpha}{2} = \frac{60^\circ + 360^\circ k}{2} \quad \text{or} \quad \frac{2\alpha}{2} = \frac{120^\circ + 360^\circ k}{2}$$

$$\boxed{\{\alpha \mid \alpha = 30^\circ + 180^\circ k \text{ or } \alpha = 60^\circ + 180^\circ k\}}$$

Example: Find all solutions to $\tan(4x) = 1$ in the interval $(0, \pi)$.

$$\tan(4x) = 1$$

$$\frac{4x}{4} = \frac{\pi/4 + k\pi}{4}$$

$$x = \pi/16 + k\pi/4$$

Start at $\pi/16$ & keep adding $\pi/4$ until we are no longer in $(0, \pi)$.

$$\boxed{\{\pi/16, 5\pi/16, 9\pi/16, 13\pi/16\}}$$

Example: Find all real number solutions to $\cos(x/2) = \sqrt{3}/2$.

$$\cos\left(\frac{x}{2}\right) = \frac{\sqrt{3}}{2}$$

$$2\left(\frac{x}{2}\right) = \left(\frac{\pi}{6} + 2k\pi\right)2 \quad \text{or} \quad 2\left(\frac{x}{2}\right) = \left(\frac{11\pi}{6} + 2k\pi\right)2$$

$$\boxed{\{x \mid x = \pi/3 + 4k\pi \text{ or } x = 11\pi/3 + 4k\pi\}}$$

Example: Find all solutions to $\csc(2x) = 2\sqrt{3}/3$ in the interval $(0^\circ, 360^\circ)$.

$$\csc(2x) = \frac{2\sqrt{3}}{3} = \frac{2}{\sqrt{3}}$$

If the csc is $\frac{2}{\sqrt{3}}$, then the sine is $\frac{\sqrt{3}}{2}$.

$$\sin(2x) = \frac{\sqrt{3}}{2}$$

$$\frac{2x}{2} = \frac{60^\circ + 360^\circ k}{2} \quad \text{or} \quad \frac{2x}{2} = \frac{120^\circ + 360^\circ k}{2}$$

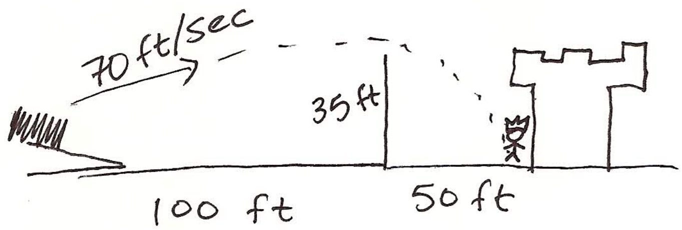
$$x = 30^\circ + 180^\circ k \quad \text{or} \quad x = 60^\circ + 180^\circ k$$

$$\boxed{\{30^\circ, 60^\circ, 210^\circ, 240^\circ\}}$$

The Path of a Projectile

The distance d (in feet) traveled by a projectile fired from the ground with an angle of elevation θ is related to the initial velocity v_0 (in ft/sec) by the equation $v_0^2 \sin 2\theta = 32d$. If the projectile is fired from the origin into the first quadrant, then the x- and y-coordinates (in feet) of the projectile at time t (in seconds) are given by $x = v_0 t \cos\theta$ and $y = -16t^2 + v_0 t \sin\theta$.

Example: A catapult is placed 100 feet from the castle wall, which is 35 feet high. A soldier wants a burning bale of hay to clear the top of the wall and land 50 feet inside the castle wall. If the initial velocity of the bale is 70 ft/sec, then at what angle should the bale of hay be launched so that it will travel 150 feet and pass over the castle wall?



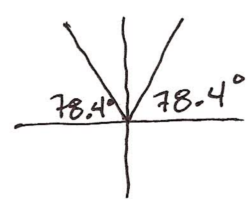
$V_0 = 70 \quad d = 150$

$V_0^2 \sin 2\theta = 32d$

$70^2 \sin 2\theta = 32(150)$

$\sin 2\theta = \dfrac{32(150)}{70^2}$

$2\theta = \sin^{-1}\left(\dfrac{32(150)}{70^2}\right) = 78.4°$



or $2\theta = 180° - 78.4° = 101.6°$

$\theta = \dfrac{78.4°}{2} = \boxed{39.2°}$ or $\theta = \dfrac{101.6°}{2} = \boxed{50.8°}$

Now we need to make sure that the burning bale of hay is at least 35 ft high when it reaches the castle wall (when it has gone 100 ft forward).

$x = V_0 t \cos\theta$

$100 = 70t \cos 39.2°$

$t = \dfrac{100}{70 \cos 39.2°} = 1.84 \text{ sec}$

$y = -16t^2 + V_0 t \sin\theta$

$y = -16(1.84)^2 + (70)(1.84)\sin 39.2°$

$y = 27.2 \text{ ft}$

If launched at a 39.2° angle, the hay will only be 27.2 ft high when it reaches the wall and will not clear the wall.

$x = V_0 t \cos\theta$

$100 = 70t \cos 50.8°$

$t = \dfrac{100}{70 \cos 50.8°} = 2.26 \text{ sec}$

$y = -16t^2 + V_0 t \sin\theta$

$y = -16(2.26)^2 + (70)(2.26)\sin 50.8°$

$y = 40.9 \text{ ft}$

If launched at a 50.8° angle, the hay will be high enough to clear the castle wall.

$\boxed{50.8°}$

Math 1060 – 4.4 Notes

Trigonometric Equations of the Quadratic Type

A General Strategy for Solving Trigonometric Equations

1. Know the solutions to $\sin x = 0$, $\cos x = 0$, and $\tan x = 0$.
2. Solve an equation involving multiple angles as if the equation had a single variable.
3. Simplify complicated equations by using identities. If possible, try to get an equation involving only one trigonometric function.
4. If possible, factor to get different trigonometric functions into separate factors.
5. For equations of the quadratic type, solve by factoring or by the quadratic formula.
6. Square each side of the equation, if necessary, so that identities involving squares can be applied. (Remember that this sometimes leads to extraneous solutions—check your answers.)

Examples: Find all real number solutions of the following equations. radians

a) $\sin x \tan x + \sin x = 0$

$\sin x(\tan x + 1) = 0$

$\sin x = 0$ or $\tan x + 1 = 0$

$0, \pi, 2\pi$, etc. $\quad \tan x = -1$

$3\pi/4, 7\pi/4$, etc.

$$\boxed{\{x \mid x = k\pi \text{ or } x = 3\pi/4 + k\pi\}}$$

b) $\sin(2x) = \cos x$

$2\sin x \cos x = \cos x$

$2\sin x \cos x - \cos x = 0$

$\cos x(2\sin x - 1) = 0$

$\cos x = 0$ or $2\sin x - 1 = 0$

$\pi/2, 3\pi/2$, etc. $\quad \sin x = 1/2$

$\pi/6, 5\pi/6$

$$\boxed{\{x \mid x = \pi/2 + k\pi \text{ or } x = \pi/6 + 2k\pi \text{ or } x = 5\pi/6 + 2k\pi\}}$$

Examples: Find all angles in $[0°, 360°)$ that satisfy the following equations.

a) $6\cos^2\left(\frac{x}{2}\right) - 7\cos\left(\frac{x}{2}\right) + 2 = 0$

$U = \cos\left(\frac{x}{2}\right)$

$6U^2 - 7U + 2 = 0$

$6(2) = 12$

Factors of 12	Sum = -7
-1, -12	-13
-2, -6	-8
-3, -4	-7

$6U^2 - 3U - 4U + 2 = 0$

$(6U^2 - 3U) + (-4U + 2) = 0$

$3U(2U - 1) - 2(2U - 1) = 0$

$(3U - 2)(2U - 1) = 0$

$U = 2/3$ or $U = 1/2$

$$\boxed{\{96.4°, 120°\}}$$

$\cos\left(\frac{x}{2}\right) = 2/3$ or $\cos\left(\frac{x}{2}\right) = 1/2$

$2\left(\frac{x}{2} = 48.2° + 360°k\right)$ $\quad 2\left(\frac{x}{2} = 60° + 360°k\right)$

or $2\left(\frac{x}{2} = 311.8° + 360°k\right)$ $\quad$ or $2\left(\frac{x}{2} = 300° + 360°k\right)$

$x = 96.4° + 720°k$ $\quad x = 120° + 720°k$

$x = 623.6° + 720°k$ $\quad x = 600° + 720°k$

b) $\cos\alpha - \sin^2\alpha = 0$

$\cos\alpha - (1 - \cos^2\alpha) = 0$

$\cos^2\alpha + \cos\alpha - 1 = 0$

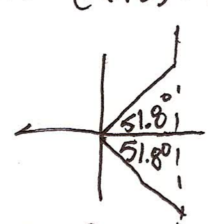
$\cos\alpha = \dfrac{-1 \pm \sqrt{1^2 - 4(1)(-1)}}{2(1)}$

$\cos\alpha = \dfrac{-1 \pm \sqrt{5}}{2}$

$\cos\alpha = .618$ or $\cos\alpha = -1.618$

$\cos^{-1}(.618) = 51.8°$

no solution
cosine is between -1 & 1.



$360° - 51.8° = 308.2°$

$$\boxed{\{51.8°, 308.2°\}}$$

Examples: Find all solutions to the following equations in the interval $[0, 2\pi)$.

a) $\left(\sin\alpha - \cos\alpha\right)^2 = \left(\frac{1}{\sqrt{2}}\right)^2$ squared both sides $\Rightarrow$ check answers

$\sin^2\alpha - 2\sin\alpha\cos\alpha + \cos^2\alpha = \frac{1}{2}$

$1 - 2\sin\alpha\cos\alpha = \frac{1}{2}$

$1 - \sin(2\alpha) = \frac{1}{2}$

$-\sin(2\alpha) = -\frac{1}{2}$

$\sin(2\alpha) = \frac{1}{2}$

$2\alpha = \pi/6 + 2k\pi$ or $2\alpha = 5\pi/6 + 2k\pi$

$\alpha = \pi/12 + k\pi$ or $\alpha = 5\pi/12 + k\pi$

$\cancel{\pi/12}, 5\pi/12, 13\pi/12, \cancel{17\pi/12}$

$\boxed{\{5\pi/12, 13\pi/12\}}$

b) $\frac{\sin(2x)}{\cos(2x)} = \frac{3\cos(2x)}{\cos(2x)}$

$\tan(2x) = 3$

$2x = 1.25 + k\pi$

$x = .625 + k\pi/2$

$\boxed{\{.625, 2.195, 3.766, 5.337\}}$

Example: Find all angles in $[0°, 360°)$ that satisfy $\cos(2x)\cos(x) - \sin(2x)\sin x = \sqrt{3}/2$.

$\cos(2x + x) = \sqrt{3}/2$

$\cos(3x) = \sqrt{3}/2$

$3x = 30° + 360°k$ or $3x = 330° + 360°k$

$x = 10° + 120°k$ or $x = 110° + 120°k$

$\boxed{\{10°, 110°, 130°, 230°, 250°, 350°\}}$

Example: Find all solutions in $[0, 2\pi)$ that satisfy $\sin x\cos(\pi/3) - \cos x\sin(\pi/3) = \sqrt{2}/2$.

$\sin(x - \pi/3) = \sqrt{2}/2$

$x - \pi/3 = \pi/4 + 2k\pi$ or $x - \pi/3 = 3\pi/4 + 2k\pi$

$x = 7\pi/12 + 2k\pi$ or $x = \frac{13\pi}{12} + 2k\pi$

$\boxed{\{7\pi/12, 13\pi/12\}}$